First draft

On the Multifractal Description of Lagrangian Field Theory

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Abstract

We show that the structure of effective field theories replicates the geometry of multifractal sets, which are built from the fundamental concepts of *scales* and *measures*. It is found that the Standard Model (SM) Lagrangian is characterized by a dominant generalized dimension $D_{SM} = 2$, while the same dimension of Einstein-Hilbert Lagrangian turns out to be $D_{GR} = 4$. On the one hand, this result disfavors any trivial unification of SM and General Relativity (GR) based on field theory. On the other, it hints that the continuous spectrum of dimensions between D_{SM} and D_{GR} may naturally account for the existence of non-baryonic Dark Matter.

1. Multifractals: a concise overview

As it is known, the *box-counting dimension* defines the main scaling property of fractal structures and is a measure of their self-similarity. Multifractals are global mixtures of fractal structures, each characterized by its local box-counting dimension. Self-similarity of multifractals is accordingly defined in terms of a *multifractal spectrum* describing the overall distribution of dimensions. In the language of chaos and complexity theory, multifractal analysis is the study of *invariant sets* and is a powerful tool for the characterization of generic *dynamical systems*.

In the recursive construction of multifractal sets from $i = 1, 2..., N$ local scales r_i with probabilities p_i, the definition of the box-counting dimension leads to [ ]

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1 \tag{1a}$$

in which

$$\sum_{i=1}^{N} p_i = 1 \tag{1b}$$

Here, q and $\tau(q)$ are two arbitrary exponents and the latter is typically presented as

$$\tau(q) = (1-q) D_q \tag{1c}$$

where D_q plays the role of a *generalized dimension*.

The closure relationship (1a) may be extended to a continuous distribution of scales in D - dimensional space time. It reads

$$\boxed{\int p^q(x)\, r^{\tau(q)}(x)\, d^D x = 1} \tag{2}$$

2. GR as topological analogue of SM

Consider now the field makeup of the SM, formed by 16 *independent* "flavors": two massive gauge bosons (W, Z), gluon (g), the Higgs scalar (H), neutrinos, charged leptons and quarks. The SM structure can be conveniently organized in the 4×4 matrix

$$SM = \begin{pmatrix} g & \nu_e & \nu_\mu & \nu_\tau \\ W & e & \mu & \tau \\ Z & u & c & b \\ H & d & s & t \end{pmatrix} \tag{3}$$

The photon (γ) is absent from (3) as it is built from the underlying components of the electroweak sector, whereby $\gamma = \gamma(W_\mu^3, B_\mu)$ and $B_\mu = B_\mu(W_\mu^3, Z)$ [ ].

It was shown in [ ] that, near the electroweak scale M_{EW}, the spectrum of particle masses m_i entering the SM satisfies the "closure" relation

$$\sum_{i=1}^{16} \left(\frac{m_i}{M_{EW}}\right)^2 = 1 \tag{4}$$

It is apparent that (3) shares the same formal structure with the metric tensor of GR, that is,

$$GR = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{5}$$

where there are only 10 independent entries under the standard assumption $g_{\mu\nu} = g_{\nu\mu}$. Starting from the GR definitions of interval and proper time leads to ($c = 1$)

$$\sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1 \tag{6}$$

subject to the constraint

$$\sum_{\nu=0}^{3} g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \tag{7}$$

Comparing (1), (4) and (6) reveals the following mapping

$$\boxed{GR: (\, p_i \Rightarrow g^{\mu\nu} g_{\nu\rho},\ q = \tfrac{1}{2}\ ,\ D_q = 4\,,\ \tau(q) = 2\,)}$$

$$\boxed{SM: (\, p_i \Rightarrow 1,\ q = 0,\ D_q = \tau(q) = 2\,)} \tag{8}$$

It is instructive to note that $D_0 = 2$ coincides with the fractal dimension of quantum mechanical paths in free space [ ], whereas $D_{1/2} = 4$ recovers the four-dimensionality of geodesic paths in classical spacetime.

A couple of conclusions may be drawn from (8):

- GR may be viewed as topological analogue of the SM, defined by a half-unitary exponent q and a dimension that is twice the SM dimension (that is, $D_{1/2} = 2D_0$).
- The spectrum of particle *mass scales* (m_i / M_{EW}) and the *four-vector of differential coordinates* ($dx^{\mu} / d\tau$) form the basis for the multifractal description of SM and GR, respectively.

3. Multifractal formulation of effective field theories

Effective Lagrangians may be described as sums of polynomial terms having the generic form

$$L(\varphi,\partial\varphi) = \sum_{i,k,l,m,n} [c_{i,i}(\partial\varphi)^k + c_{i,i+1}(\partial\varphi)^l(\varphi)^m + c_{i+1,i+1}(\varphi)^n] \tag{8}$$

To simplify notation, we focus below on the basic unit entering the sum (8), namely on

$$L_u(\varphi,\partial\varphi) = c_{11}(\partial\varphi)^k + c_{12}(\partial\varphi)^l(\varphi)^m + c_{22}(\varphi)^n \Rightarrow c_{11}z_1^k + c_{12}z_1^l z_2^m + c_{22}z_2^n = 1 \tag{9}$$

in which

$$z_1^k = \frac{(\partial\varphi)^k}{L(\varphi,\partial\varphi)},\ z_1^l = \frac{(\partial\varphi)^l}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^m = \frac{\varphi^m}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^n = \frac{\varphi^n}{L(\varphi,\partial\varphi)} \tag{10}$$

$c_{1,2,3}$ are constants at given setting, for example, at a given energy scale. Therefore,

$$\boxed{r_{11}^k + r_{12}^l r_{21}^m + r_{22}^n = 1} \tag{11}$$

where

$$r_{11}^k = c_{11}z_1^k,\ r_{12}^l = \sqrt{c_{12}}\, z_1^l,\ r_{21}^m = \sqrt{c_{12}}\, z_2^m,\ r_{22}^n = c_{22}z_2^n$$

If $c_{1,2,3}$ depend on the field content or their derivatives, (9) assumes the general form

$$\boxed{c_{11}^{q_1}(r_{11})\, r_{11}^k + c_{12}^{q_2}(r_{12}, r_{21}) r_{12}^l r_{21}^m + c_{22}^{q_3}(r_{22}) r_{22}^n = 1} \tag{12}$$

where $q_{1,2,3}$ are non-vanishing exponents and

$$c_{11} + c_{12} + c_{22} = 1 \tag{13}$$

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4. GR as multifractal set

Einstein-Hilbert action:

$$S = \int R\sqrt{-g}\, d^4x \tag{14}$$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \tag{15}$$

$$g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma})\sqrt{-g} = 2L_G\sqrt{-g} \tag{16}$$

$$L_G = \frac{dS}{\sqrt{-g}d^4x} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \to g^{\mu\nu}\frac{\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}}{L_G} = 1 \tag{17}$$

$$\sum_{\nu=0}^{3} g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \tag{18}$$

5. SM as multifractal set

SM Lagrangian

$$L_{SM} = -\frac{1}{4}\sum_{V} V^{a}_{\mu\nu}V^{a\mu\nu} + \overline{f^{i}_{L}}i\gamma^{\mu}D_{\mu}f^{i}_{L} + \overline{f^{i}_{R}}i\gamma^{\mu}D_{\mu}f^{i}_{R} + (Y_{ij}\overline{f^{i}_{L}}Hf^{j}_{R} + h.c.) + (D^{\mu}H)^{\dagger}(D_{\mu}H) - V(H) \tag{19}$$

Here, the summation convention over repeated indices is assumed, with $(i, j) = 1,2,3$ extending over the three fermion families [ ]. The vector fields V corresponds to the three gauge groups of the SM, namely $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$,

$$V = \left\{B, W^{a=1,2,3}, G^{a=1\ldots8}\right\} \tag{20}$$

to which we associate the field-strength tensors

$$V^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g\, f_{abc} V^b_\mu V^c_\nu \tag{21}$$

and covariant derivative operators

$$D_\mu = \partial_\mu - i \sum_V g_V\, t^a_V V^a_\mu \tag{22}$$

The last couple of terms denote the kinetic and potential contributions of the Higgs field,

$$V(H) = -m^2_H\, H^\dagger H + \lambda (H^\dagger H)^2 \tag{23}$$

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Appendix

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Quantum Mechanics	Standard Model	General Relativity
Invariance under changes in representation	Invariance under local gauge transformations	Diffeomorphism invariance

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References

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